A Theory of Higher-Order Subtyping with Type Intervals

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ICPF 2021 – 22–27 Aug 2021

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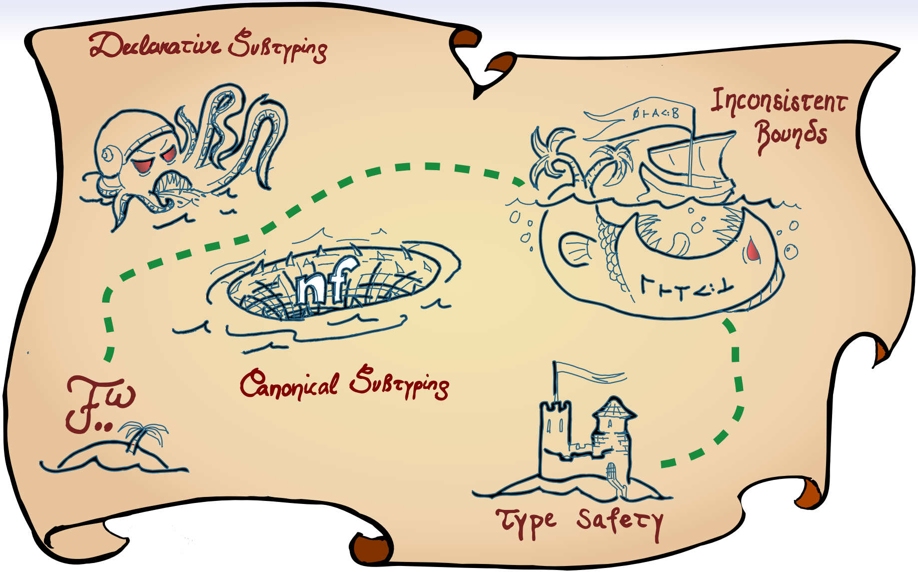
Declarative Subtyping
Inconsistent Bounds
$\emptyset \vdash A <: B$
$\Gamma \vdash \top <: \bot$
nf
$F^{\omega}_{..}$
Canonical Subtyping
Type Safety

DOT and Dotty

DOT

WadlerFest, April 2016

The Essence of Dependent Object Types

Nada Amin[1], Samuel Grütter[1], Martin Odersky[1]( ), Tiark Rompf[2],
and Sandro Stucki[1]

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Abstract. Focusing on path-dependent types, the paper develops foundations for Scala from first principles. Starting from a simple calculus $D_{<:}$ of dependent functions, it adds records, intersections and recursion to arrive at DOT, a calculus for dependent object types. The paper shows an encoding of System F with subtyping in $D_{<:}$ and demonstrates the expressiveness of DOT by modeling a range of Scala constructs in it.

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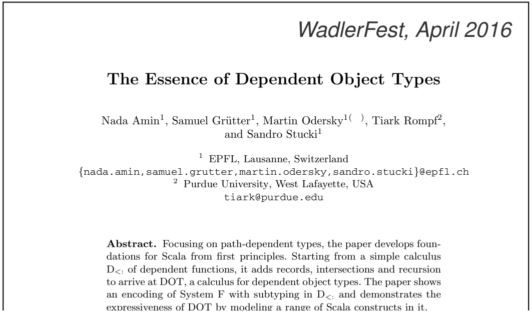
- a minimal core calculus for Scala



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Dotty/Scala 3

Scala Symposium, Oct 2016

Implementing Higher-Kinded Types in Dotty

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proved to be challenging, so much so that we evaluated four different strategies before settling on the current direct representation encoding. The strategies are summarized as follows:

- A *simple encoding* in the DOT-inspired [9] core type structures that can express partial applications and not much more
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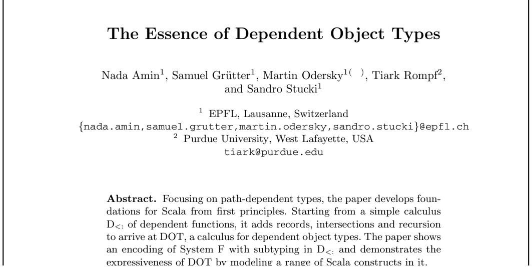

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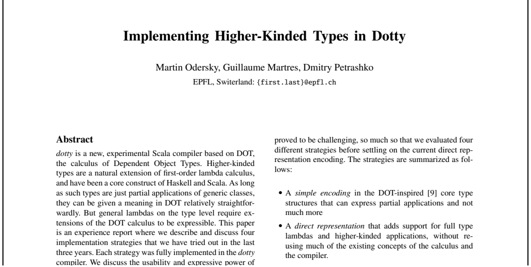

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HK Types – An Example

```
type Ordering[A] = (A, A) => Boolean

abstract class SortedView[A, B >: A](xs: List[A], ord: Ordering[B]) {
  def foldLeft[C](z: C, op: (C, A) => C): C
  def concat[C >: A <: B](ys: List[C]): SortedView[C, B]
  // declarations of further operations such as ’map’, ’flatMap’, etc.
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- Type parameters of operators can also have bounds!
- Type definitions can be used to introduce aliases.

The Anatomy of a Type Interval

```
X >: A <: B
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Intuition: X has bounds $A <: X <: B$.

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Intuition: X is an element of the set of types $\{ A <: \cdots <: B \}$

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Special cases

Upper bound `X <: B` $\qquad X : \bot \mathinner{..} B$

- $\bot =$ `Nothing` $=$ minimal/bottom type;

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Upper bound	`X <: B`	$X : \bot \mathrel{..} B$
Lower bound	`X >: A`	$X : A \mathrel{..} \top$

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Alias	`X = A`	$X : A \mathrel{..} A$

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- $\bot \mathrel{..} \top = * =$ kind of all types.
- $A \mathrel{..} A =$ singleton containing only A.

The Anatomy of a Type Interval (cont.)

```
F[X >: A <: B] >: G <: H
```

We can also represent bounded operators

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`F[X >: A <: B] >: G <: H` $\qquad F : (X{:}A\,..\,B) \to G\,..\,H$

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Examples

Alias	`F1[X] = List[X]`	$F_1 : (X{:}*) \to \mathsf{List}\,X \,..\, \mathsf{List}\,X$

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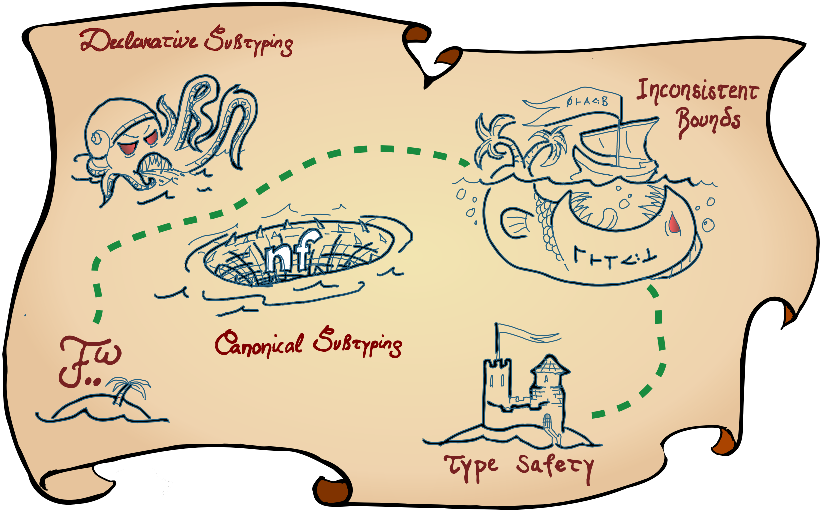
NB. The operators $F_1 - F_3$ all have dependent kinds.

Proving Type Safety of $F^{\omega}_{\cdot\cdot}$



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Problem: $\beta\eta$-conversions get in the way of inversion.

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Solution: normalize types and kinds – no redexes, no conversions!

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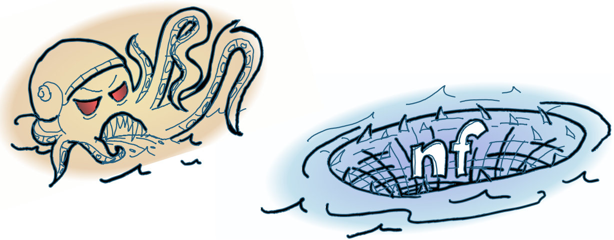

New problem: dependent kinding of applications involves substitutions.

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New solution: use hereditary substitution

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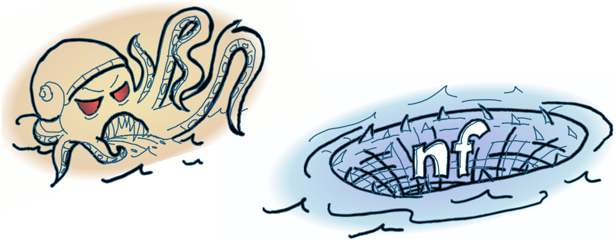

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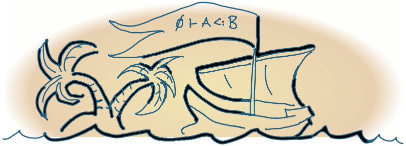
$$\frac{\Gamma \vdash Z : (X{:}J) \to K \qquad \Gamma \vdash V : J}{\Gamma \vdash Z\,V : K[V/X^{|J|}]}$$

New solution: use hereditary substitution (introducing further problems. . .)

Challenge 3: Inconsistent Bounds

Problem: Type variables can introduce arbitrary subtyping relationships.



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NB. This causes all sorts of problems:

- subject reduction (preservation) fails,
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Solution: invert $<:$ only for closed types
– no variables, no inconsistencies!

Inversion – Step by Step

declarative

$$\varnothing \vdash_{\mathsf{d}} A \to B <: A' \to B'$$

Inversion – Step by Step

declarative canonical

$$\varnothing \vdash_{\mathsf{d}} A \to B <: A' \to B' \xrightarrow{\mathsf{nf}} \varnothing \vdash_{\mathsf{c}} U \to V <: U' \to V'$$

- $U = \mathsf{nf}(A)$, $V = \mathsf{nf}(B)$, ...

Inversion – Step by Step

declarative | canonical | transitivity-free

$$\varnothing \vdash_{\mathsf{d}} A \to B <: A' \to B' \xrightarrow{\mathsf{nf}} \varnothing \vdash_{\mathsf{c}} U \to V <: U' \to V' \xrightarrow{\simeq} \vdash_{\mathsf{tf}} U \to V <: U' \to V'$$

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Inversion – Step by Step

declarative canonical transitivity-free

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$$\big\downarrow \text{invert}$$

$$\vdash_{\mathsf{tf}} U' <: U$$
$$\vdash_{\mathsf{tf}} V <: V'$$

- $U = \mathsf{nf}(A)$, $V = \mathsf{nf}(B)$, ...

Inversion – Step by Step

$$
\begin{tikzcd}
\text{declarative} & \text{canonical} & \text{transitivity-free} \\
\varnothing \vdash_{\mathsf{d}} A \to B <: A' \to B' \arrow[r, "\mathsf{nf}"] & \varnothing \vdash_{\mathsf{c}} U \to V <: U' \to V' \arrow[r, "\simeq"] & \vdash_{\mathsf{tf}} U \to V <: U' \to V' \arrow[d, "\text{invert}"] \\
& \begin{array}{l} \varnothing \vdash_{\mathsf{c}} U' <: U \\ \varnothing \vdash_{\mathsf{c}} V \ <: V' \end{array} & \begin{array}{l} \vdash_{\mathsf{tf}} U' <: U \\ \vdash_{\mathsf{tf}} V \ <: V' \end{array} \arrow[l, "\simeq"']
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\begin{array}{l} \varnothing \vdash_{\mathsf{d}} A' = U' <: U = A \\ \varnothing \vdash_{\mathsf{d}} B = V <: V' = B' \end{array} & \begin{array}{l} \varnothing \vdash_{\mathsf{c}} U' <: U \\ \varnothing \vdash_{\mathsf{c}} V <: V' \end{array} \arrow[l, "\text{nf sound}"'] & \begin{array}{l} \vdash_{\mathsf{tf}} U' <: U \\ \vdash_{\mathsf{tf}} V <: V' \end{array} \arrow[l, "\simeq"']
\end{tikzcd}
$$

- $U = \mathsf{nf}(A)$, $V = \mathsf{nf}(B)$, ...
- nf sound: $\Gamma \vdash A = \mathsf{nf}_\Gamma(A)$ for all Γ and A.

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. . . and in the artifact (`https://zenodo.org/record/5060213`).

- Mechanization of the full metatheory!

Thank you!

Coauthor

Paolo Giarrusso

Collaborators

- Guillaume Martres
- Nada Amin
- Martin Odersky
- Andreas Abel
- Jesper Cockx

Check out the Agda mechanization!

`https://github.com/sstucki/f-omega-int-agda`
`https://zenodo.org/record/5060213`