A Theory of Higher-Order Subtyping with Type Intervals

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DOT and Dotty

WadlerFest, April 2016

The Essence of Dependent Object Types

Nada Amin¹, Samuel Grutter¹, Martin Odersky¹( ), Tiark Rompf²,
and Sandro Stucki³

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Abstract. Focusing on path-dependent types, the paper develops foundations for Scala from first principles. Starting from a simple calculus $D_c$ of dependent functions, it adds records, intersections and recursion to arrive at DOT, a calculus for dependent object types. The paper shows an encoding of System $F$ with subtyping in $D_c$ and demonstrates the expressiveness of DOT by modeling a range of Scala constructs in it.
DOT and Dotty

DOT

• a minimal core calculus for Scala

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- a minimal core calculus for Scala
- proven type-safe (in Coq)

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• does not support HK types

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Dotty/Scala 3

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Implementing Higher-Kinded Types in Dotty

Martin Odersky, Guillaume Martres, Dmitry Petrashko
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Abstract
dotty is a new, experimental Scala compiler based on DOT, the calculus of Dependent Object Types. Higher-kinded types are a natural extension of first-order lambda calculus, and have been a core construct of Haskell and Scala. As long as such types are just partial applications of generic classes, they can be given a meaning in DOT relatively straightforward. But general lambdas on the type level require extensions of the DOT calculus to be expressible. This paper is an experience report where we describe and discuss four implementation strategies that we have tried out in the last three years. Each strategy was fully implemented in the dotty compiler. We discuss the usability and expressive power of implemented sound support for these higher-kinded types in Dotty and have been a core construct of Haskell and Scala. As long as such types are just partial applications of generic instance, they can be given a meaning in DOT relatively straightforward. But general lambdas on the type level require extensions of the DOT calculus to be expressible. This paper is an experience report where we describe and discuss four implementation strategies that we have tried out in the last three years. Each strategy was fully implemented in the dotty compiler. We discuss the usability and expressive power of

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Scala Symposium, Oct 2016

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Dotty/Scala 3

• a Scala compiler based on DOT

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proved to be challenging, so much so that we evaluated four different strategies before settling on the current direct representation encoding. The strategies are summarized as follows:

• A simple encoding in the DOT-inspired [9] core type structures that can express partial applications and not much more
• A direct representation that adds support for full type lambdas and higher-kind applications, without re-using much of the existing concepts of the calculus and the compiler.
DOT and Dotty

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Dotty/Scala 3

- a Scala compiler based on DOT
- type safety unclear

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S. Stucki, P. G. Giarrusso

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HK Types – An Example

type Ordering[A] = (A, A) => Boolean

abstract class SortedView[A, B >: A](xs: List[A], ord: Ordering[B]) {
    def foldLeft[C](z: C, op: (C, A) => C): C
    def concat[C >: A <: B](ys: List[C]): SortedView[C, B]
    // declarations of further operations such as 'map', 'flatMap', etc.
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• Type parameters of operators can also have bounds!
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```scala
// An example of a higher-order type.

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  def foldLeft[C](z: C, op: (C, A) => C): C
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```

- Types can take parameters: i.e. we have type operators.
- Type parameters of methods can have bounds (as usual).
- Type parameters of operators can also have bounds!
- Type definitions can be used to introduce aliases.
The Anatomy of a Type Interval

\[ X \geq A \leq B \]
The Anatomy of a Type Interval

\[ X >: A <: B \]

*Intuition:* \( X \) has bounds \( A <: X <: B \).
The Anatomy of a Type Interval

\[ X >: A <: B \]

*Intuition:* \( X \) is an element of the set of types \( \{ A <: \cdots <: B \} \)
The Anatomy of a Type Interval

\[ X >: A <: B \]

Intuition: \( X \) is an element of the set of types \( \{ A <: \cdots <: B \} = A .. B \)
The Anatomy of a Type Interval

\[ X >: A <: B \]

\[ X : A .. B \]

*Intuition:* \( X \) is an element of the set of types \( \{ A <: \cdots <: B \} = A .. B \)
The Anatomy of a Type Interval

\[ X >: A <: B \quad X : A .. B \]

**Intuition:** \( X \) is an element of the set of types \( \{ A <: \cdots <: B \} = A .. B \)

**Special cases**

- **Upper bound** \( X <: B \quad X : \perp .. B \)
- \( \perp = \text{Nothing} = \text{minimal/bottom type}; \)
- \( \top = \text{Any} = \text{maximal/top type}; \)
- \( \ast = \text{kind of all types.} \)
- \( A .. A = \text{singleton containing only } A. \)

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The Anatomy of a Type Interval

\[ X >: A <: B \quad \quad X : A .. B \]

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**Special cases**

- **Upper bound** \( X <: B \quad X : \perp .. B \)
- **Lower bound** \( X >: A \quad X : A .. \top \)

- \( \perp = \text{Nothing} = \text{minimal/bottom type} \)
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The Anatomy of a Type Interval

\[ X >: A <: B \quad X : A .. B \]

*Intuition:* \( X \) is an element of the set of types \( \{ A <: \cdots <: B \} = A .. B \)

**Special cases**

- **Upper bound** \( X <: B \quad X : \bot .. B \)
- **Lower bound** \( X >: A \quad X : A .. \top \)
- **Abstract** \( X \quad X : \bot .. \top \)

- \( \bot = \text{Nothing} = \text{minimal/bottom type}; \)
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The Anatomy of a Type Interval

\[ X \geq A \leq B \quad X : A .. B \]

**Intuition:** \( X \) is an element of the set of types \( \{ A <: \cdots <: B \} = A .. B \)

**Special cases**

- **Upper bound**
  \[ X <: B \quad X : \bot .. B \]

- **Lower bound**
  \[ X >: A \quad X : A .. \top \]

- **Abstract**
  \[ X \quad X : \bot .. \top \]

- **Alias**
  \[ X = A \quad X : \bot .. A \]

- \( \bot = \text{Nothing} = \text{minimal/bottom type} \)
- \( \top = \text{Any} = \text{maximal/top type} \)
- \( \bot .. \top = \ast = \text{kind of all types} \)
- \( A .. A = \text{singleton containing only } A \)
The Anatomy of a Type Interval (cont.)

\[ F[X >: A <: B] >: G <: H \]

We can also represent bounded operators
The Anatomy of a Type Interval (cont.)

\[ F[X >: A <: B] >: G <: H \]

We can also represent bounded operators

\[ F : (X:A..B) \rightarrow G..H \]
The Anatomy of a Type Interval (cont.)

\[ F[X >: A <: B] >: G <: H \quad F : (X:A .. B) \to G .. H \]

We can also represent **bounded operators**

Examples

- **Alias**
  
  \[ F_1[X] = \text{List}[X] \]
  
  \[ F_1 : (X:* \to \text{List} X .. \text{List} X \]

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The Anatomy of a Type Interval (cont.)

\[ F[X >: A <: B] >: G <: H \quad F : (X:A .. B) \rightarrow G .. H \]

We can also represent bounded operators

Examples

- **Alias**
  \[ F1[X] = \text{List}[X] \quad F_1 : (X:* \rightarrow \text{List} X .. \text{List} X \]

- **Upper bound**
  \[ F2[X] <: \text{List}[X] \quad F_2 : (X:* \rightarrow \bot .. \text{List} X \]

NB. The operators \( F_1 - F_3 \) all have dependent kinds.
The Anatomy of a Type Interval (cont.)

\[ F[X >: A <: B] >: G <: H \]
\[ F : (X:A .. B) \to G .. H \]

We can also represent bounded operators

Examples

- **Alias**
  \[ F_1[X] = \text{List}[X] \]
  \[ F_1 : (X:* \to \text{List} X .. \text{List} X) \]

- **Upper bound**
  \[ F_2[X] <: \text{List}[X] \]
  \[ F_2 : (X:* \to \bot .. \text{List} X) \]

- **HO bounded op.**
  \[ F_3[X, Y[_ <: X]] \]
  \[ F_3 : (X:* \to (Y:\bot .. X) \to *) \to * \]
The Anatomy of a Type Interval (cont.)

\[ F[X >: A <: B] >: G <: H \]

\[ F : (X:A .. B) \to G .. H \]

We can also represent bounded operators

**Examples**

- **Alias**
  \[ F1[X] = \text{List}[X] \]
  \[ F_1 : (X:* ) \to \text{List} X .. \text{List} X \]

- **Upper bound**
  \[ F2[X] <: \text{List}[X] \]
  \[ F_2 : (X:* ) \to \bot .. \text{List} X \]

- **HO bounded op.**
  \[ F3[X, Y[_ <: X]] \]
  \[ F_3 : (X:* ) \to (Y:(_:\bot .. X) \to *) \to * \]

**NB.** The operators \( F_1 \) – \( F_3 \) all have dependent kinds.
Proving Type Safety of $F_\omega$.

Main sub-challenges:
1. Subtyping derivations may involve computation (βη-conversions).
2. Subtyping derivations may involve subsumption (via subkinding).
3. Type variables with inconsistent bounds can reflect arbitrary subtyping assumptions into subtyping derivations.

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A Theory of Higher-Order Subtyping with Type Intervals
Proving Type Safety of $F^\omega$.

The big challenge is to prove subtyping inversion.
Proving Type Safety of $F^\omega$.

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\[
\Gamma \vdash A_1 \rightarrow B_1 <: A_2 \rightarrow B_2 : *
\]
\[
\Gamma \vdash A_2 <: A_1 : *
\quad \Gamma \vdash B_1 <: B_2 : *
\]

\[
\Gamma \vdash \forall X:K_1. A_1 <: \forall X:K_2. A_2 : *
\]
\[
\Gamma \vdash K_2 <: K_1
\quad \Gamma, X:K_2 \vdash A_1 <: A_2 : *
\]
The big challenge is to prove subtyping inversion.

\[
\Gamma \vdash A_1 \rightarrow B_1 <: A_2 \rightarrow B_2 : *
\]
\[
\Gamma \vdash A_2 <: A_1 : * \quad \Gamma \vdash B_1 <: B_2 : *
\]

Main sub-challenges:

1. Subtyping derivations may involve computation (\(\beta\eta\)-conversions).
Proving Type Safety of $F^\omega$.

The big challenge is to prove subtyping inversion.

\[
\begin{align*}
\Gamma \vdash A_1 \rightarrow B_1 <: A_2 \rightarrow B_2 : * \\
\Gamma \vdash A_2 <: A_1 : * & \quad \Gamma \vdash B_1 <: B_2 : * \\
\Gamma \vdash \forall X : K_1. A_1 <: \forall X : K_2. A_2 : * \\
\Gamma \vdash K_2 <: K_1 & \quad \Gamma, X : K_2 \vdash A_1 <: A_2 : *
\end{align*}
\]

Main sub-challenges:

1. Subtyping derivations may involve computation ($\beta\eta$-conversions).
2. Subtyping derivations may involve subsumption (via subkinding).
The big challenge is to prove subtyping inversion.

\[
\Gamma \vdash A_1 \rightarrow B_1 <: A_2 \rightarrow B_2 : *
\]

\[
\frac{}{\Gamma \vdash A_2 <: A_1 : * \quad \Gamma \vdash B_1 <: B_2 : *}
\]

\[
\frac{}{\Gamma \vdash \forall X: K_1. A_1 <: \forall X: K_2. A_2 : *}
\]

\[
\frac{}{\Gamma \vdash K_2 <: K_1 \quad \Gamma, X: K_2 \vdash A_1 <: A_2 : *}
\]

Main sub-challenges:

1. Subtyping derivations may involve computation (\(\beta\eta\)-conversions).
2. Subtyping derivations may involve subsumption (via subkinding).
3. Type variables with inconsistent bounds can reflect arbitrary subtyping assumptions into subtyping derivations.
The big challenge is to prove subtyping inversion.

\[
\frac{\Gamma \vdash A_1 \rightarrow B_1 <: A_2 \rightarrow B_2 : *}{\Gamma \vdash A_2 <: A_1 : * \quad \Gamma \vdash B_1 <: B_2 : *}
\]

\[
\frac{\Gamma \vdash \forall X : K_1. A_1 <: \forall X : K_2. A_2 : *}{\Gamma \vdash K_2 <: K_1 \quad \Gamma, X : K_2 \vdash A_1 <: A_2 : *}
\]
Challenge 1: Getting Rid of $\beta\eta$-Conversions

Problem: $\beta\eta$-conversions get in the way of inversion.

\[
\Gamma \vdash A_1 \rightarrow A_2 <: (\lambda X:* . X \rightarrow A_2)A_1 <: \cdots <: (\lambda X:* . X \rightarrow B_2)B_1 <: B_1 \rightarrow B_2 : *
\]
Challenge 1: Getting Rid of $\beta\eta$-Conversions

Problem: $\beta\eta$-conversions get in the way of inversion.

$\Gamma \vdash A_1 \rightarrow A_2 <: (\lambda X : *. X \rightarrow A_2) A_1 <: \cdots <: (\lambda X : *. X \rightarrow B_2) B_1 <: B_1 \rightarrow B_2 : *$

Solution: normalize types and kinds – no redexes, no conversions!
Challenge 1: Getting Rid of $\beta\eta$-Conversions

New problem: dependent kinding of applications involves substitutions.

\[
\Gamma \vdash Z : (X : J) \rightarrow K \quad \Gamma \vdash V : J
\]

\[
\Gamma \vdash ZV : K[V/X]
\]
Challenge 1: Getting Rid of $\beta\eta$-Conversions

New problem: dependent kinding of applications involves substitutions.

\[
\frac{\Gamma \vdash Z : (X:J) \rightarrow K \quad \Gamma \vdash V : J}{\Gamma \vdash Z \, V : K[V/X]}
\]
Challenge 1: Getting Rid of $\beta\eta$-Conversions

New problem: dependent kinding of applications involves substitutions.

\[ \Gamma \vdash Z : (X : J) \rightarrow K \quad \Gamma \vdash V : J \]

\[ \Gamma \vdash Z \, V : K[V/X] \]

New solution: use hereditary substitution
Challenge 1: Getting Rid of $\beta\eta$-Conversions

New problem: dependent kinding of applications involves substitutions.

\[
\Gamma \vdash Z : (X : J) \rightarrow K \quad \Gamma \vdash V : J
\]
\[
\Gamma \vdash ZV : K[V/X][J]
\]

New solution: use hereditary substitution (introducing further problems. . .)
Challenge 3: Inconsistent Bounds

**Problem:** Type variables can introduce *arbitrary* subtyping relationships.

\[
\vdash A \rightarrow B <: \top <: X <: \bot <: \forall Y : K. C :
\]

NB. This causes all sorts of problems:

- subject reduction (preservation) fails,
- subtyping becomes undecidable,
- ...

**Solution:** invert `<` only for closed types – no variables, no inconsistencies!
Challenge 3: Inconsistent Bounds

Problem: Type variables can introduce inconsistent subtyping relationships.
Challenge 3: Inconsistent Bounds

Problem: Type variables can introduce inconsistent subtyping relationships.

\[
X: \top \ldots \bot \vdash \quad X : * \quad \text{NB. This causes all sorts of problems:}
\]

- subject reduction (preservation) fails,
- subtyping becomes undecidable,
  ...

Solution: invert \(\prec\) only for closed types – no variables, no inconsistencies!

S. Stucki, P. G. Giarrusso
A Theory of Higher-Order Subtyping with Type Intervals
Challenge 3: Inconsistent Bounds

Problem: Type variables can introduce inconsistent subtyping relationships.

\[
X : \top \ldots \perp \vdash \top <: X <: \bot <: \forall Y : K . C
\]

NB. This causes all sorts of problems:
• subject reduction (preservation) fails,
• subtyping becomes undecidable,
• . . .

Solution: invert \(<\) only for closed types – no variables, no inconsistencies!
Challenge 3: Inconsistent Bounds

Problem: Type variables can introduce inconsistent subtyping relationships.

\[
X : \top \ldots \perp \vdash A \rightarrow B \leq : \top \leq : X \quad : * 
\]

NB. This causes all sorts of problems:

• subject reduction (preservation) fails,
• subtyping becomes undecidable,

Solution: invert \( \leq \) only for closed types – no variables, no inconsistencies!

S. Stucki, P. G. Giarrusso
A Theory of Higher-Order Subtyping with Type Intervals
Challenge 3: Inconsistent Bounds

Problem: Type variables can introduce inconsistent subtyping relationships.

\[
X: \top \ldots \bot \vdash A \to B \hypsub \top \hypsub X \hypsub \bot \quad : * 
\]

NB. This causes all sorts of problems:

• subject reduction (preservation) fails,
• subtyping becomes undecidable,
• . . .

Solution: invert \hypsub only for closed types – no variables, no inconsistencies!
Challenge 3: Inconsistent Bounds

**Problem:** Type variables can introduce **inconsistent** subtyping relationships.

\[
X: \top \ldots \bot \vdash A \rightarrow B <: \top <: X <: \bot <: \forall Y: K. C : *
\]

NB. This causes all sorts of problems:
- subject reduction (preservation) fails,
- subtyping becomes undecidable,
- ...

**Solution:** invert `<` only for closed types – no variables, no inconsistencies!
Challenge 3: Inconsistent Bounds

Problem: Type variables can introduce inconsistent subtyping relationships.

\[ X : \top \ldots \bot \vdash A \rightarrow B <: \top <: X <: \bot <: \forall Y : K. C : * \]

NB. This causes all sorts of problems:
- subject reduction (preservation) fails,
- subtyping becomes undecidable,
- …
Challenge 3: Inconsistent Bounds

Problem: Type variables can introduce inconsistent subtyping relationships.

\[
X : \top \ldots \perp \vdash A \rightarrow B <: \top <: X <: \perp <: \forall Y : K. C : \ast
\]

NB. This causes all sorts of problems:

- subject reduction (preservation) fails,
- subtyping becomes undecidable,
- …

Solution: invert $<$: only for closed types
– no variables, no inconsistencies!
Inversion – Step by Step

declarative

\[ \emptyset \vdash_d A \rightarrow B <: A' \rightarrow B' \]
Inversion – Step by Step

declarative  canonical

$$\emptyset \vdash_d A \rightarrow B <: A' \rightarrow B' \xrightarrow{\text{nf}} \emptyset \vdash_c U \rightarrow V <: U' \rightarrow V'$$

• $$U = \text{nf}(A), \ V = \text{nf}(B), \ldots$$
Inversion – Step by Step

declarative  canonical  transitivity-free

\[ \emptyset \vdash_{d} A \to B <: A' \to B' \xrightarrow{\text{nf}} \emptyset \vdash_{c} U \to V <: U' \to V' \xrightarrow{\sim} \vdash_{tf} U \to V <: U' \to V' \]

- \( U = \text{nf}(A), \ V = \text{nf}(B) \), \ldots
Inversion – Step by Step

declarative

\[ \emptyset \vdash_d A \rightarrow B <: A' \rightarrow B' \quad \text{nf} \]

canonical

\[ \emptyset \vdash_c U \rightarrow V <: U' \rightarrow V' \quad \sim \]

transitivity-free

\[ \vdash_{tf} U \rightarrow V <: U' \rightarrow V' \]

\[ \vdash_{tf} U' <: U \]

\[ \vdash_{tf} V <: V' \]

\[ \text{invert} \]

\[ \bullet U = \text{nf}(A), \ V = \text{nf}(B), \ldots \]
Inversion – Step by Step

declarative  canonical  transitivity-free

\[ \emptyset \vdash_d A \rightarrow B <: A' \rightarrow B' \xrightarrow{\text{nf}} \emptyset \vdash_c U \rightarrow V <: U' \rightarrow V' \xrightarrow{\sim} \vdash_{\text{tf}} U \rightarrow V <: U' \rightarrow V' \]

\[ \emptyset \vdash_c U' <: U \]

\[ \emptyset \vdash_c V <: V' \xrightarrow{\sim} \vdash_{\text{tf}} V <: V' \]

\[ U = \text{nf}(A), \ V = \text{nf}(B), \ldots \]
Inversion – Step by Step

- **declarative**
  \[ \emptyset \vdash_d A \rightarrow B <: A' \rightarrow B' \]

- **canonical**
  \[ \emptyset \vdash_c U \rightarrow V <: U' \rightarrow V' \xrightarrow{\sim} \vdash_{tf} U \rightarrow V <: U' \rightarrow V' \]

- **transitivity-free**
  \[ \emptyset \vdash_d A' = U' <: U = A \]
  \[ \emptyset \vdash_c V' = B' \]

- **nf sound**
  \[ \emptyset \vdash_c U' <: U \]
  \[ \emptyset \vdash_c V' <: V' \xrightarrow{\sim} \vdash_{tf} V' <: V' \]

- **nf sound**
  \[ \Gamma \vdash A = \text{nf}_\Gamma (A) \text{ for all } \Gamma \text{ and } A. \]

- **U = nf(A), V = nf(B), ...**
There’s More in the Paper…

• Recap of the $\mathcal{F}_\omega$ family and high-level intro to $\mathcal{F}_\omega$· (with examples).
• Full presentation of $\mathcal{F}_\omega$· (syntax, typing, SOS, . . .).
• Undecidability of subtyping.
• Additional definitions and lemmas.
• Human-readable proofs for (most) results.
• Mechanization of the full metatheory!


. . . and in the artifact ([https://zenodo.org/record/5060213](https://zenodo.org/record/5060213)).
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https://arxiv.org/abs/2107.01883

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• Mechanization of the full metatheory!
Thank you!

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- Nada Amin
- Martin Odersky
- Andreas Abel
- Jesper Cockx

Check out the Agda mechanization!

https://github.com/sstucki/f-omega-int-agda
https://zenodo.org/record/5060213